Int. J. Solids Structures Vol. 21, No. 6, pp. 561-572, 1985
Printed in the U.S.A.

0020-7683/85 $3.00 + .00
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BEHAVIOUR OF VISCOELASTIC-VISCOPLASTIC SPHERES AND CYLINDERS—FULLY PLASTIC VESSEL WALLS

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(*Received* 20 *June* 1983; *in revised form* 23 *July* 1984)

Abstract—The material model consists of a viscoelastic Burgers element and an additional viscoplastic Bingham element when the effective stress exceeds the yield stress. For fully plastic vessel walls, exact closed-form expressions are derived for the stress and strain state in pressurised or relaxation loaded thick-walled cylinders in plane strain and spheres. For the spherical problem, the material compressibility is accounted for. The influence of the different material parameters on the behaviour of the vessels is evaluated. It is shown that the magnitude of the Maxwell viscosity is of major importance for the long-term behaviour of thick-walled fully plastic vessels.

INTRODUCTION

Typically, the creep behaviour of materials changes from a linear dependence for small stresses to a nonlinear stress dependence for higher loadings. A combined viscoelastic-viscoplastic model can simulate such a material behaviour. Below a certain yield stress, a linear viscoelastic behaviour occurs, whereas additional viscoplastic response occurs if this yield stress is exceeded. Adopting such a viscoelastic-viscoplastic material model, the quasi-static time-dependent behaviour of partly plastic, thick-walled spheres and cylinders was treated in [1]. Here we shall consider the case in which the vessel walls are fully plastic.

Madejski[2] treated a fully plastic sphere loaded by a constant inner pressure, adopting a simplified elastic-viscoplastic behaviour. Wierzbicki[3], on the other hand, included a viscous behaviour below the yield stress through the use of a Maxwell element and treated the behaviour of a pressurised sphere as well as one under relaxation conditions. The loadings were assumed to be constant with time.

Compared with these previous results, the solutions derived in this study represent significant extensions. First, incompressible cylinders in plane strain and spheres are treated in a unified fashion. Second, pressurised as well as relaxation loaded vessels are treated, and these loadings might vary with time. In addition, we apply a quite general constitutive model consisting of a viscoelastic Burgers model below the yield stress as well as a viscoplastic Bingham model above this limit. For the spherical problem, the material compressibility is accounted for.

For this spectrum of problems, we shall present exact, closed-form solutions of the stress and strain fields, and a detailed discussion will be given with emphasis on the principal aspects of the vessel behaviour and the influence of the different material parameters.

CONSTITUTIVE MODELLING

The material is assumed to have a creep-sensitive deviatoric response, while its volumetric response is purely elastic. Below a certain yield stress, we assume a viscoelastic response corresponding to a Burgers material, i.e. a Maxwell and a Kelvin element in series. Above the yield stress, we assume an additional viscoplastic response corresponding to a Bingham element. The material response is symbolized in Fig. 1.

It appears that the material model reflects many creep characteristics that can be observed for a variety of materials. Owing to the friction element, the creep behaviour

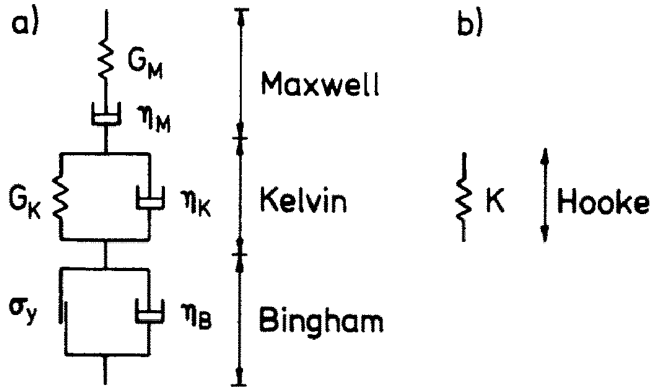

Fig. 1. Material model: (a) deviatoric response, (b) volumetric response.

depends nonlinearly on the stresses, and whereas the Maxwell and the Bingham elements exhibit secondary, irreversible creep, the Kelvin element exhibits primary, reversible creep. The presence of the Kelvin element is therefore important for many applications. However, its inclusion complicates the calculations considerably as it increases the order of the involved differential equations from one to two.

The deviatoric stress and strain tensor are defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}, \qquad e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}, \tag{1, 2}$$

where σ_{ij} and ϵ_{ij} denote the stress and strain tensor, respectively, and where usual tensor notation is applied. Tension and elongation are considered positive.

For the Maxwell, Kelvin and Bingham elements, all quantities are labelled with the index M, K and B, respectively. Moreover, the constants G and η denote, in general, a shear modulus and a viscosity coefficient, respectively. Small strains are assumed.

The constitutive equation for the Maxwell element is

$$e_{ij,M} = \frac{s_{ij}}{2G_M} + \frac{1}{2\eta_M}\int s_{ij}\,\mathrm{d}t. \tag{3}$$

The constitutive equation for the Kelvin element is

$$s_{ij} = 2\eta_K \dot{e}_{ij,K} + 2G_K e_{ij,K}, \tag{4}$$

where a dot denotes the time derivative. For stresses below the yield stress, the Bingham element is rigid; otherwise, the constitutive equation for the Bingham element is

$$e_{ij,B} = \frac{1}{2\eta_B}\int\left(1 - \frac{\sigma_y}{\sigma_e}\right)s_{ij}\,\mathrm{d}t, \quad \text{for} \quad \sigma_e \geqslant \sigma_y, \tag{5}$$

where σ_y is the yield stress and σ_e is the effective stress of von Mises defined by

$$\sigma_e = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2}. \tag{6}$$

Contrary to the viscous deviatoric response as defined above, the volumetric behaviour is assumed to be purely elastic, i.e.

$$\epsilon_{ii} = \frac{\sigma_{ii}}{3K}, \tag{7}$$

where the constant K is the bulk modulus.

UNIFIED FORMULATION FOR SPHERES AND CYLINDERS

For spheres, only two directions are of interest. It follows from (6) that

$$\sigma_e = T(\sigma_\theta - \sigma_r) \tag{8}$$

in familiar notation. Depending on the problem, we choose $T = 1$ or $T = -1$ so as to ensure that the effective stress is positive. To attain closed-form solutions for cylinders, it becomes necessary to assume incompressibility whereby $\epsilon_{ij} = e_{ij}$. Moreover, only plane strain is considered, i.e. $\epsilon_z = 0$. As shown in [1], these assumptions imply that $\sigma_z = (\sigma_r + \sigma_\theta)/2$ and thereby

$$\sigma_e = T\frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r), \tag{9}$$

where, depending on the problem, we choose again $T = 1$ or $T = -1$ to ensure that the effective stress is positive.

To facilitate the exposition, the unified treatment of the spherical and cylindrical problems presented in [1] is clearly preferable. According to [1], the equilibrium equation can be written as

$$\frac{\partial \sigma_r}{\partial r} = \lambda \frac{\sigma_e}{r}, \tag{10}$$

where

$$\lambda = \begin{cases} 2T, & \text{for spherical problems} \\ \dfrac{2T}{\sqrt{3}}, & \text{for cylindrical problems.} \end{cases} \tag{11}$$

Moreover, the circumferential deviatoric strain can be written as

$$e_\theta = \frac{f(t)}{r^\alpha} - M\,\sigma_e, \tag{12}$$

where

$$\alpha = \begin{cases} 3, & \text{for spherical problems} \\ 2, & \text{for cylindrical problems} \end{cases} \tag{13}$$

and where

$$M = \begin{cases} \dfrac{2T}{9K}, & \text{for spherical problems} \\ 0, & \text{for cylindrical problems.} \end{cases} \tag{14}$$

Note that the function $f(t)$ depends only on time, whereas incompressibility, i.e. $K \to \infty$, implies that $M = 0$.

Denoting the inner radius by r_1 and the outer one by r_2, the boundary conditions are

$$r = r_2, \qquad \sigma_r = -p_2(t) \tag{15}$$

$$r = r_1, \qquad \begin{cases} \sigma_r = -p_1(t), & \text{stress boundary problem} \\ u = u_1(t), & \text{displacement boundary problem.} \end{cases} \tag{16}$$

That is, we will consider pressurised vessels as well as relaxation of vessels. However, even for the displacement boundary problem, there exists a pressure along the inner surface, and this so-called shrink-fit pressure varies with time. Therefore, the boundary condition (16) suggests that

$$r = r_1, \qquad \sigma_r = -p_1(t) \tag{17}$$

always applies. For relaxation problems, the shrink-fit pressure is still unknown. It will appear that it can be determined using (16). It is assumed that the initial load is applied suddenly; i.e. an elastic state exists at $t = 0^+$.

We shall also state expressions for the circumferential and radial strain. According to [1], the circumferential strain, defined by $\epsilon_\theta = u/r$ where u is the radial displacement, can be written in the following unified way:

$$\epsilon_\theta = \frac{f(t)}{r^\alpha} + \frac{3M}{2T}\sigma_r. \tag{18}$$

Similarly, the radial strain becomes

$$\epsilon_r = -(\alpha - 1)\frac{f(t)}{r^\alpha} + \frac{3M}{2T}(2T\sigma_e + \sigma_r). \tag{19}$$

Note that for incompressible materials, where $M = 0$ [see (14)], the strains depend on the radius through the factor $r^{-\alpha}$ alone, irrespective of the material model.

PRELIMINARY EXPRESSIONS FOR THE STRESSES AND STRAINS

Having presented the constitutive equations, the initial and boundary conditions as well as some general expressions for the vessels, we are now in a position to derive some preliminary expressions for the effective stress and the time-dependent function $f(t)$ present, for instance, in (12). These expressions are identical to those derived in [1]. Consequently, the constitutive equation can be written as

$$\ddot{\sigma}_e + A\,\dot{\sigma}_e + B\,\sigma_e = C\frac{\ddot{f}}{r^\alpha} + D\frac{\dot{f}}{r^\alpha} + F, \tag{20}$$

where

$$A = \frac{G_M}{1 + 3\lambda M\,G_M}\left(\frac{1}{\eta_M} + \frac{1}{\eta_K} + \frac{1}{\eta_B}\right) + \frac{G_K}{\eta_K} \tag{21}$$

$$B = \frac{G_M\,G_K}{(1 + 3\lambda M\,G_M)\eta_K}\left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right) \tag{22}$$

$$C = \frac{3\lambda\,G_M}{1 + 3\lambda M\,G_M} \tag{23}$$

$$D = \frac{3\lambda\,G_M\,G_K}{(1 + 3\lambda M\,G_M)\eta_K} \tag{24}$$

$$F = \frac{G_M\,G_K\,\sigma_y}{(1 + 3\lambda M\,G_M)\eta_K\,\eta_B} \tag{25}$$

and where the term λM is always nonnegative. The constitutive equation (20) applies always so long as the material point is in a viscoelastic-viscoplastic state.

Let us now assume that the material point initially is also in a viscoelastic-viscoplastic state. The initial load is assumed to be applied suddenly at $t = 0^+$, and the resulting two initial conditions become (see [1])

$$\sigma_{e,0} = C\frac{f_0}{r^\alpha} \tag{26}$$

and

$$\dot{\sigma}_{e,0} = C\frac{\dot{f}_0}{r^\alpha} - (AC - D)\frac{f_0}{r^\alpha} + N, \tag{27}$$

where

$$N = \frac{G_M\,\sigma_y}{(1 + 3\lambda M\,G_M)\,\eta_B}. \tag{28}$$

The roots of the characteristic equation belonging to (20) are

$$\left.\begin{matrix} R'\ (\leq 0) \\ R''\ (< 0) \end{matrix}\right\} = \frac{1}{2}\left(-A \pm \sqrt{A^2 - 4B}\right). \tag{29}$$

Trivial considerations show that $A^2 - 4B \geqslant 0$ always holds. In accordance with [1], we can write the solution to (20) subjected to conditions (26) and (27) as

$$\sigma_e = \Phi(t) + \frac{Cf(t) + \psi(t)}{r^\alpha}, \tag{30}$$

where the function $\Phi(t)$ is defined by

$$\Phi(t) = \frac{e^{R't} - e^{R''t}}{R' - R''}\left(N + \frac{\sigma_y\,R''}{1 + \eta_B/\eta_M}\right) + \frac{\sigma_y}{1 + \eta_B/\eta_M}(1 - e^{R''t}). \tag{31}$$

The unknown function $\psi(t)$ present in (30) is given by

$$\psi(t) = -\frac{1}{R' - R''}\left[R''\,(CR'' + D)\,e^{R''t}\int_0^t f(t)\,e^{-R''t}\,dt - R'\,(CR' + D)\,e^{R't}\int_0^t f(t)\,e^{-R't}\,dt\right]. \tag{32}$$

Note that solution (30) applies to material points that initially are in a viscoelastic-viscoplastic state and that remain in such a state. The fundamental property of (30) is that the effective stress depends on radius and time through separated functions. By letting $\eta_B \to \infty$ in the parameterrs defined above, the expressions degenerate to those valid for a viscoelastic material.

The solution above determines the effective stress as a function of $f(t)$. Alternatively, (20) subjected to conditions (26) and (27) can be solved to determine $f(t)$ as a function of the effective stress. As shown in [1], this solution becomes

$$\frac{f(t)}{r^\alpha} = \frac{1}{3\lambda}\left[\frac{3\lambda}{C}\sigma_e + \frac{e^{(-G_K/\eta_K)t}}{\eta_K}\int_0^t \sigma_e\,e^{(G_K/\eta_K)t}\,dt + \left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right)\int_0^t \sigma_e\,dt - \frac{\sigma_y}{\eta_B}t\right]. \tag{33}$$

VISCOPLASTIC ZONE SPREADS ALL THROUGH THE VESSEL WALL

The present treatment of viscoelastic-viscoplastic vessels where the viscoplastic zone spreads all through the vessel wall turns out to be much simpler than when a viscoelastic zone exists (see [1]). This follows from the absence of the delicate problem of a moving plastic boundary.

As the whole vessel wall is in a viscoplastic state, solution (30) applies all through the vessel wall. Inserting (30) into equilibrium equation (10) and integrating from r_1 to r_2 using boundary conditions (15) and (17) enables us to determine an expression for the unknown term $Cf(t) + \psi(t)$ present in (30). Inserting this expression into (30) yields, after trivial calculations, the following closed-form solution for the effective stress:

$$\sigma_e = \Phi(t)\left[1 - \frac{\ln(r_2/r_1)^\alpha}{r^\alpha\,(1/r_1^\alpha - 1/r_2^\alpha)}\right] + \frac{\alpha(p_1 - p_2)}{\lambda\, r^\alpha\,(1/r_1^\alpha - 1/r_2^\alpha)}, \tag{34}$$

where $\Phi(t)$ is given by (31). Inserting (34) in equilibrium equation (10) and integrating from r_1 to r provides the following closed-form solution for the radial stress:

$$\sigma_r = -p_1 + (p_1 - p_2)\frac{1/r_1^\alpha - 1/r^\alpha}{1/r_1^\alpha - 1/r_2^\alpha} + \lambda\Phi(t)\left(\ln\frac{r}{r_1} - \frac{1/r_1^\alpha - 1/r^\alpha}{1/r_1^\alpha - 1/r_2^\alpha}\ln\frac{r_2}{r_1}\right). \tag{35}$$

Note that these solutions are valid for arbitrary outer pressures $p_1(t)$ and $p_2(t)$ so long as the vessel wall remains fully plastic.

Suppression of the viscoplastic behaviour is obtained by letting $\eta_B \to \infty$. This implies that the parameter N given by (28) becomes 0. That is, (31) becomes

$$\Phi(t)_{\eta_B\to\infty} = 0. \tag{36}$$

Using this expression in (34) and (35), we rediscover the familiar elastic and viscoelastic solutions.

If the Kelvin element is ignored, i.e. $\eta_K \to \infty$, the parameter B given by (22) becomes 0, i.e. $R' \to 0$ and $R'' \to -A = -N(1 + \eta_B/\eta_M)/\sigma_y$ [see (29)]. The function $\Phi(t)$ given by (31) then becomes

$$\Phi(t)_{\eta_K\to\infty} = \frac{\sigma_y}{1 + \eta_B/\eta_M}\,[1 - e^{-N/\sigma_y\,(1+\eta_B/\eta_M)t}], \tag{37}$$

and (34) and (35) reduce to the expressions derived by Wierzbicki[3] for a sphere loaded by a constant inner pressure.

If the Maxwell element is also ignored, i.e. $\eta_M \to \infty$, we obtain the response of a simple elastic-viscoplastic material, and (37) degenerates to

$$\Phi(t)_{\eta_K\to\infty,\eta_M\to\infty} = \sigma_y(1 - e^{-(N/\sigma_y)t}). \tag{38}$$

In this case, (34) and (35) become equivalent to the solutions derived by Madejski[2] for a sphere loaded by a constant inner pressure.

Consider now the stress field for large times. Equation (31) shows that

$$\Phi(t)_{t\to\infty} = \frac{\sigma_y}{1 + \eta_B/\eta_M}. \tag{39}$$

That is, the effective stress expression (34) becomes

$$\sigma_{e,t\to\infty} = \frac{\sigma_y}{1 + \eta_B/\eta_M}\left[1 - \frac{\ln(r_2/r_1)^\alpha}{r^\alpha(1/r_1^\alpha - 1/r_2^\alpha)}\right] + \frac{\alpha(p_1 - p_2)}{\lambda r^\alpha(1/r_1^\alpha - 1/r_2^\alpha)}. \tag{40}$$

It is of interest to note that this expression, valid for large times, holds even when the pressure difference $p_1 - p_2$ varies. That is, after some time, there will be no stress redistribution due to time effects. The stress changes are caused by pressure changes alone. For constant outer pressures, we therefore approach a stationary stress field given by (40). It appears that the only material parameters that influence the stress state given by (40) are σ_y and the η_B/η_M ratio. If $\eta_B/\eta_M \to \infty$, then the stress field becomes equal to the linear elastic distribution. It is of interest that $\eta_B/\eta_M \to \infty$ can occur both when $\eta_B \to \infty$ (suppression of viscoplastic behaviour) and when $\eta_M \to 0$ (very creep-sensitive Maxwell element). The Kelvin parameters have no influence on the stationary stress state. The same observations were achieved for the stationary stress field for pressurised vessels having a viscoelastic zone (see [1]). The development with time of the effective stress for a cylinder with constant outer pressures is illustrated in Fig. 2.

Let us now return to expression (34). It appears that the effective stress depends on the outer pressures through the last term alone, whereas the function $\phi(t)$ present in the first term is responsible for the redistribution with time of the stress field. By setting the factor to $\phi(t)$ equal to 0, we can therefore determine a radius, r_c, for which the effective stress depends on the outer pressures alone. This yields

$$\left(\frac{r_c}{r_2}\right)^\alpha = \frac{\ln(r_2/r_1)^\alpha}{(r_2/r_1)^\alpha - 1}. \tag{41}$$

As $r_2/r_1 > 1$, (41) implies that $r_c/r_2 < 1$ and $r_c/r_1 > 1$; i.e. the radius r_c for which the effective stress depends on the outer loadings alone exists always. It is of interest that the r_c value is independent of the material parameters as well as the loading. For the particular case of constant outer pressures, these conclusions are in agreement with those of Wierzbicki's treatment of a sphere without the Kelvin element[3]. Figure 2 demonstrates the significance of the r_c radius. The effective stress at the constant r_c radius becomes

$$\sigma_{ec} = \frac{\alpha(p_1 - p_2)}{\lambda\, r_c^\alpha\,(1/r_1^\alpha - 1/r_2^\alpha)} \tag{42}$$

where r_c is given by (41).

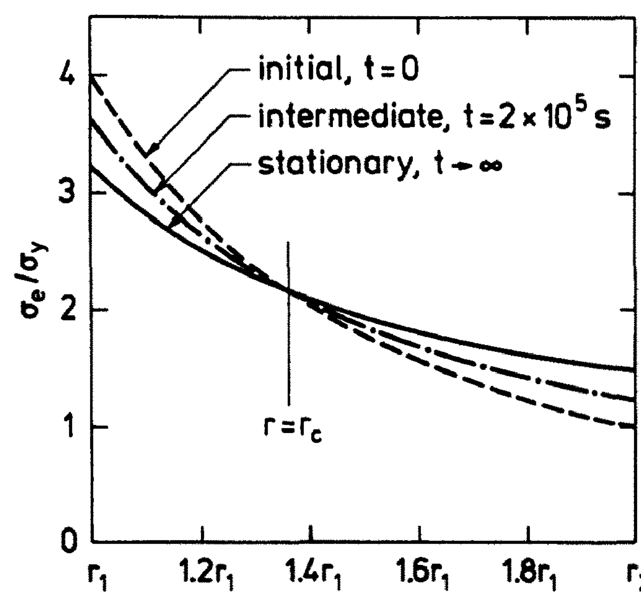

Fig. 2. Effective stress fields for fully plastified pressurized cylinder: $r_1/r_2 = 1/2$, $(p_1 - p_2)/\sigma_y = \sqrt{3}$, $G_M = 5000$ MPa, $\eta_M = 10^{10}$ MPa·sec, $G_K = 2000$ MPa·sec, $\eta_K = \eta_B = 10^9$ MPa, $M = 0$.

Consider now the particular case where the outer pressures are constant. The effective stress at the inner boundary takes its maximum value at time $t = 0$. As the effective stress is a steadily decreasing function of radius and as it is constant at $r = r_c$, this suggests that the effective stress at the outer boundary takes its minimum value at time $t = 0$. This has the important consequence that a vessel loaded by constant outer pressures will remain fully plastified if it initially is so. Figure 2 illustrates this point.

The existence of the fixed radius r_c, where the effective stress depends on the outer pressures alone, is identical to the existence of a "skeletal" point as defined by Marriott[4]. Determination of the strain field requires knowledge of the $f(t)$ function [see (18) and (19)]. This function can be determined by means of (33) using the general effective stress expression (34). However, knowledge of the existence of a skeleton point and the corresponding effective stress facilitates this determination greatly. Use of (42) and (41) in (33) yields the result directly:

$$f(t) = \frac{\alpha}{3\lambda^2(1/r_1^\alpha - 1/r_2^\alpha)}\left[\frac{3\lambda(p_1 - p_2)}{C} + \frac{e^{(-G_K/\eta_K)t}}{\eta_K} \pm \int_0^t (p_1 - p_2)e^{(G_K/\eta_K)t}\,dt + \left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right)\int_0^t (p_1 - p_2)\,dt - t\frac{\lambda\sigma_y}{\eta_B}\ln\frac{r_2}{r_1}\right]. \tag{43}$$

This expression completes the full determination of the stress and strain fields in pressurised vessels. For relaxation loaded vessels, we still remain to determine the shrink-fit pressure $p_1(t)$.

Let us evaluate some simplified behaviours. If the outer pressures are constant with time, (43) reduces to

$$f(t) = \frac{\alpha}{3\lambda^2(1/r_1^\alpha - 1/r_2^\alpha)}\left((p_1 - p_2)\left\{\frac{3\lambda}{C} + \frac{1}{G_K}[1 - e^{(-G_K/\eta_K)t}] + \left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right)t\right\} - t\frac{\lambda\sigma_y}{\eta_B}\ln\frac{r_2}{r_1}\right). \tag{44}$$

If this expression is used to determine the circumferential strain given by (18) and if the Kelvin element is suppressed, i.e. $\eta_K \to \infty$, the resulting circumferential strain becomes identical to that derived by Wierzbicki[3] for a sphere loaded by constant outer pressures. If the viscoplastic behaviour is suppressed, i.e. $\eta_B \to \infty$, we obtain the response of a Burgers material. In this case, (44) becomes identical to the solution given by Gnirk and Johnson[5] for the cylindrical problem.

For vessels loaded by constant outer pressures, it was shown previously that a stationary stress state exists. From (18) follows then $\dot{\epsilon}_\theta = \dot{f}/r^\alpha$, and differentiation of (44) provides

$$\dot{\epsilon}_{\theta,t\to\infty} = \frac{\alpha}{3\lambda^2 r^\alpha(1/r_1^\alpha - 1/r_2^\alpha)}\left[(p_1 - p_2)\left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right) - \frac{\lambda\sigma_y}{\eta_B}\ln\frac{r_2}{r_1}\right]. \tag{45}$$

That is, the stationary stress state results in a stationary strain rate state. This stationary strain rate is independent of the elastic parameters, and as the Kelvin element is now rigid, neither of the Kelvin parameters influences it.

The response of pressurised vessels has now been determined completely, and a discussion of principal issues has been given. The results are valid for changing outer pressures so long as the vessel remains fully plastified. The results obtained are also valid for relaxation loaded vessels, but in this case the shrink-fit pressure $p_1(t)$ is still unknown. The determination of this pressure is the subject of the next section.

Relaxation loaded vessels

We shall now consider the relaxation problem, and in particular we shall determine the unknown shrink-fit pressure $p_1(t)$ from boundary condition (16). The outer pressure $p_2(t)$ is a known quantity.

Evaluating the circumferential strain (18) at the inner boundary and making use of (43) give, after some rearrangement,

$$\frac{3\lambda^2}{\alpha}\left[1-\left(\frac{r_1}{r_2}\right)^{\alpha}\right]\left(\frac{u_1}{r_1}+\frac{3M}{2T}p_2\right)+t\frac{\lambda\sigma_y}{\eta_B}\ln\frac{r_2}{r_1}$$
$$=\left(\frac{1}{G_M}+3\lambda M\left\{1-\frac{3\lambda}{2T\alpha}\left[1-\left(\frac{r_1}{r_2}\right)^{\alpha}\right]\right\}\right)(p_1-p_2)$$
$$+\frac{e^{(-G_K/\eta_K)t}}{\eta_K}\int_0^t(p_1-p_2)\,e^{(G_K/\eta_K)t}\,dt+\left(\frac{1}{\eta_M}+\frac{1}{\eta_K}\right)\int_0^t(p_1-p_2)\,dt, \quad (46)$$

where the relation $3\lambda/C = 1/G_M + 3\lambda M$ has been used [see (23)]. This integral equation is used to determine the unknown pressure $p_1(t)$. However, it becomes convenient to transform (46) into the corresponding differential equation.

Observing that $M = 0$ for the cylinder and $3\lambda/(2T\alpha) = 1$ for the sphere, the coefficient to the term $p_1 - p_2$ can be written as

$$\frac{1}{G_M}+3\lambda M\left\{1-\frac{3\lambda}{2T\alpha}\left[1-\left(\frac{r_1}{r_2}\right)^{\alpha}\right]\right\}=\frac{1}{G_M}+3\lambda M\left(\frac{r_1}{r_2}\right)^{\alpha}. \quad (47)$$

Then, differentiation of (46) and subsequent multiplication by the factor $e^{t(G_K/\eta_K)}$ yield

$$\frac{3\lambda^2}{\alpha}\left[1-\left(\frac{r_1}{r_2}\right)^{\alpha}\right]\left(\frac{\dot u_1}{r_1}+\frac{3M}{2T}\dot p_2\right)e^{(G_K/\eta_K)t}+\frac{\lambda\sigma_y}{\eta_B}e^{(G_K/\eta_K)t}\ln\frac{r_2}{r_1}$$
$$=\left[\frac{1}{G_M}+3\lambda M\left(\frac{r_1}{r_2}\right)^{\alpha}\right](\dot p_1-\dot p_2)\,e^{(G_K/\eta_K)t}-\frac{G_K}{\eta_K^2}\int_0^t(p_1-p_2)\,e^{(G_K/\eta_K)t}\,dt$$
$$+\left(\frac{1}{\eta_M}+\frac{1}{\eta_K}+\frac{1}{\eta_B}\right)(p_1-p_2)\,e^{(G_K/\eta_K)t}. \quad (48)$$

Differentiation of (48) gives

$$\ddot p_1-\ddot p_2+I(\dot p_1-\dot p_2)+J(p_1-p_2)$$
$$=Q\left(\frac{\ddot u_1}{r_1}+\frac{3M}{2T}\ddot p_2\right)+V\left(\frac{\dot u_1}{r_1}+\frac{3M}{2T}\dot p_2\right)+W, \quad (49)$$

where

$$I=\frac{1/\eta_M+1/\eta_K+1/\eta_B}{1/G_M+3\lambda M\,(r_1/r_2)^{\alpha}}+\frac{G_K}{\eta_K} \quad (50)$$

$$J=\frac{G_K/\eta_K\,(1/\eta_M+1/\eta_B)}{1/G_M+3\lambda M\,(r_1/r_2)^{\alpha}} \quad (51)$$

$$Q=\frac{3\lambda^2\,[1-(r_1/r_2)^{\alpha}]}{\alpha\,[1/G_M+3\lambda M\,(r_1/r_2)^{\alpha}]} \quad (52)$$

$$V=\frac{3\lambda^2\,G_K\,[1-(r_1/r_2)^{\alpha}]}{\alpha\eta_K\,[1/G_M+3\lambda M\,(r_1/r_2)^{\alpha}]} \quad (53)$$

$$W=\frac{\lambda\,\sigma_y\,G_K\ln r_2/r_1}{\eta_B\,\eta_K\,[1/G_M+3\lambda M\,(r_1/r_2)^{\alpha}]} \quad (54)$$

The roots of the characteristic equation belonging to (49) are

$$\left.\begin{matrix} s' \ (\leq 0) \\ s'' \ (<0) \end{matrix}\right\} = \frac{1}{2}(-I \pm \sqrt{I^2 - 4J}). \tag{55}$$

Trivial considerations show that $I^2 - 4J \geq 0$ always holds. The initial conditions for differential equation (49) shall now be derived. From (46), evaluated at $t = 0^+$, and by making use of (47), we obtain

$$p_{10} - p_{20} = Q\left(\frac{u_{10}}{r_1} + \frac{3M}{2T}p_{20}\right). \tag{56}$$

This expression corresponds to the trivial linear elastic solution. From (48), evaluated at $t = 0^+$, we obtain

$$\dot{p}_{10} - \dot{p}_{20} = Q\left(\frac{\dot{u}_{10}}{r_1} + \frac{3M}{2T}\dot{p}_{20}\right) - (QI - V)\left(\frac{u_{10}}{r_1} + \frac{3M}{2T}p_{20}\right) + Z, \tag{57}$$

where (56) has been applied and where we have defined

$$Z = \frac{\lambda\, \sigma_y \ln r_2/r_1}{\eta_B\,[1/G_M + 3\lambda M\,(r_1/r_2)^\alpha]}. \tag{58}$$

It appears that differential equation (49) with initial conditions (56) and (57) is completely similar to eqns (20), (26) and (27). Therefore, using (30)–(32) and observing that the term $\sigma_y/(1 + \eta_B/\eta_M)$ present in (31) is equal to F/B, we can write the solution to (49) directly as

$$\begin{aligned} p_1 - p_2 = {} & \frac{e^{s't} - e^{s''t}}{s' - s''}\left(Z + \frac{W}{J}s''\right) + \frac{W}{J}(1 - e^{s''t}) + Q\left(\frac{u_1}{r_1} + \frac{3M}{2T}p_2\right) \\ & - \frac{1}{s' - s''}\left[s''\,(Qs'' + V)\,e^{s''t}\int_0^t \left(\frac{u_1}{r_1} + \frac{3M}{2T}p_2\right)e^{-s''t}\,dt\right. \\ & \left. - s'\,(Qs' + V)\,e^{s't}\int_0^t \left(\frac{u_1}{r_1} + \frac{3M}{2T}p_2\right)e^{-s't}\,dt\right]. \end{aligned} \tag{59}$$

This expression determines the unknown shrink-fit pressure $p_1(t)$ in the very general case where $p_2(t)$ and $u_1(t)$ change with time and where the material compressibility (for the sphere) is accounted for. A simple numerical scheme in general is necessary to solve (59).

To determine the strains as given by (18) and (19), the function $f(t)$ must be evaluated knowing now the shrink-fit pressure $p_1(t)$. This function is easily determined from the displacement $u_1(t)$ given at the inner boundary. Equation (37) yields

$$f(t) = \left(\frac{u_1}{r_1} + \frac{3M}{2T}p_1\right)r_1^\alpha. \tag{60}$$

The relaxation behaviour of vessels plastified all through the vessel wall has now been determined completely for the general loading case. The following is devoted to some simplified loading situations and to an evaluation of the influence of the material parameters.

A particular simple closed-form expression of (59) can be obtained if the term $u_1/r_1 + 3Mp_2/(2T)$ is a constant. In that case, (59) reduces to

$$\begin{aligned} p_1 - p_2 = {} & \frac{e^{s't} - e^{s''t}}{s' - s''}\left(Z + \frac{W}{J}s''\right) + \frac{W}{J}(1 - e^{s''t}) \\ & - \frac{u_1/r_1 + (3M/2T)\,p_2}{s' - s''}\,[(Qs'' + V)\,e^{s''t} - (Qs' + V)\,e^{s't}]. \end{aligned} \tag{61}$$

Two situations fulfill the requirement that $u_1/r_1 + 3Mp_2/(2T)$ be a constant. In both cases, the prescribed displacement u_1 must be constant. In the first case, material compressibility is accounted for, but the pressure p_2 must be constant. In the other case, incompressibility is assumed, i.e. $M = 0$, and $p_2(t)$ is allowed to vary arbitrarily. These two situations, for which (61) applies, cover most applications.

If the Kelvin element is suppressed, i.e. $\eta_K \to \infty$, then $J = 0$, i.e. $s' = 0$ and $s'' = -I$, and a further simplification of (61) occurs. If the viscoplastic behaviour is suppressed, i.e. $\eta_B \to \infty$, we obtain the response of a Burgers material. In this case, $W = Z = 0$ holds, and if we put $M = 0$, (61) becomes identical to the expression derived previously in [1].

Let us now evaluate the response for large times. In this situation, (61) reduces to $p_1 - p_2 = W/J$. Using (51) and (54), we obtain

$$(p_1 - p_2)_{t\to\infty} = \frac{\lambda\,\sigma_y}{1 + \eta_B/\eta_M} \ln\frac{r_2}{r_1}. \tag{62}$$

This pressure limit applies also when $s' = 0$, which occurs for $\eta_K \to \infty$. Using the limit value in the expression for the long-term effective stress (40), we get

$$\sigma_{e,t\to\infty} = \frac{\sigma_y}{1 + \eta_B/\eta_M} \tag{63}$$

throughout the vessel wall. Consequently, if $\eta_B/\eta_M > 0$, the relaxation indicates that a viscoelastic zone eventually will develop, and the following response will be determined by the combined behaviour of the viscoplastic and viscoelastic zones.

If, however, the Maxwell viscosity is suppressed, i.e. $\eta_M \to \infty$, the pressure limit will be $p_1 - p_2 = \lambda\sigma_y \ln r_2/r_1$, and from (63) the long-term effective stress becomes $\sigma_e = \sigma_y$ throughout the vessel wall. That is, the relaxation behaviour approaches a stationary stress state as well as a stationary strain state with rigid Bingham and Kelvin elements. Therefore, the relaxation behaviour changes dramatically whether or not Maxwell viscosity is considered. Similar conclusions were derived previously for relaxation loaded vessels having a viscoelastic zone (see [1]).

CONCLUSIONS

Exact, closed-form solutions for the viscoelastic-viscoplastic behaviour of fully plastic, thick-walled cylinders in plane strain and spheres were derived in a unified fashion for the stress and strain fields. The loading was due either to pressurisation or relaxation. Similar solutions were presented in [1] for partly plastic vessels, but the solutions there were much impeded by the presence of a moving plastic boundary. However, it appears that much of the analysis in [1] applies also for fully plastic vessels, indicating that a unified treatment of partly and fully plastic vessels can be followed fairly extensively.

For deviatoric loading, the constitutive model consists of a viscoelastic Burgers element in combination with a viscoplastic Bingham element. For volumetric loading, linear elastic behaviour was assumed for the spherical problem, whereas incompressibility was assumed for the cylindrical one. The Kelvin part of the Burgers element exhibits primary, reversible creep, whereas both the Maxwell part of the Burgers element as well as the Bingham element exhibit secondary, irreversible creep. Thus, the constitutive model reflects many creep characteristics observed in a variety of materials. In addition, the outer pressures or the prescribed inner displacement might vary with time. This in connection with the rather general constitutive model opens for a variety of applications.

A major effort was given to a discussion of principal aspects of the vessel behaviour and of the influence of the different material parameters. It was shown that for a hierarchy of simplified material behaviours, the derived closed-form solutions degenerate to previously known results.

For vessels where the plastic zone spreads all through the vessel wall, there exists a skeleton point, i.e. a fixed radius for which the effective stress depends only on the prescribed outer loadings. This aspect facilitates greatly the analysis of such vessels.

For the pressurised vessels, a stationary stress and strain rate situation arises for large times. The magnitude of the Maxwell viscosity turns out to be of major importance for these stationary states. If vessels loaded by constant outer pressures are initially fully plastified, they will remain so.

For relaxation loaded vessels, the Maxwell viscosity is also of decisive importance. If this viscosity is ignored, the relaxation behaviour may approach a stationary stress and strain state. If Maxwell viscosity is included, the vessels will always experience a continued relaxation.

Therefore, the long-term behaviour of fully plastic vessels depends to a large extent on the magnitude of the Maxwell viscosity, whereas the Kelvin parameters are of importance only for the transient phase. It is of interest that the same conclusion applies also for partly plastic vessels (see [1]).

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